

Geometric and Stochastic Clusters of Gravitating Potts Models

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Abstract

We consider the fractal dimensions of critical clusters occurring in configurations of a q -state Potts model coupled to the planar random graphs of the dynamical triangulations formulation of Euclidean quantum gravity in two dimensions. For regular lattices, it is well-established that at criticality the properties of Fortuin-Kasteleyn clusters are directly related to the conventional critical exponents, whereas the corresponding properties of the geometric clusters of like spins are not. Recently it has been observed that the latter are related to the critical properties of a tricritical Potts model with the same central charge. We apply the KPZ formalism to develop a related prediction for the case of Potts models coupled to quantum gravity and employ numerical simulation methods to confirm it for the Ising case $q = 2$.

Key words: Potts model, Ising model, quantum gravity, Fortuin-Kasteleyn representation, fractal dimensions, annealed disorder, cluster algorithms

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1 Introduction

The relation between the percolation problem and thermal phase transitions of lattice spin systems has been a question of intense research for at least three

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decades. Clusters of even spins are natural objects occurring in the analysis of phase ordering processes and nucleation [1], and a theory of critical phenomena in terms of purely geometrical objects appears appealing. In this context, it had long been surmised that a continuous phase transition of a spin system might be accompanied (or, in fact, caused) by a percolation transition of the clusters of like spins (*geometric clusters*), the appearance of a percolating cluster sustaining the onset of a non-zero magnetisation [2]. While for the special case of the Potts model in two dimensions it turned out that, indeed, the thermal phase transition point coincides with the percolation transition of the spin clusters, this behaviour is not generic and does not occur in three-dimensional systems [3]. Also, the critical exponents associated to the percolation of geometric clusters are not directly related to the thermal exponents of the spin model [4]. However, a close relation between the percolation and thermal phase transitions can be established by considering *stochastically* defined clusters (or droplets) as they occur in the Fortuin-Kasteleyn (FK) representation of the Potts model, and it can be shown that, in fact, the Potts model is equivalent to a site-bond correlated percolation problem [5] such that the corresponding critical exponents agree. This identification of the proper cluster objects (*FK clusters*) percolating at the thermal phase transition subsequently also allowed for the design of *cluster algorithms* for the efficient simulation of Potts models in the vicinity of the ordering transition [6, 7], beating the observed critical slowing down of local update algorithms. Similarly, relations of continuous-spin models to percolation problems could be established and corresponding cluster algorithms formulated [7, 8], such that the continuous phase transitions of many standard models of statistical mechanics are by now understood in terms of the percolation properties of some suitably defined ensemble of stochastic clusters.

Although not in the universality class of the thermal phase transition, the clusters of like spins or geometric clusters still undergo a percolation transition in the course of thermal phase ordering. This transition is in general not equivalent to ordinary (site or bond) percolation [4] and it remains an interesting open question to determine the general critical behaviour of clusters of aligned spins. For the case of the two-dimensional Ising model, it has been conjectured and numerically verified that the geometric clusters are described by the $q = 1$ tricritical Potts model [9]; this correspondence can be understood from a direct construction starting from the dilute Potts model [10]. Subsequently, analogous conjectures for the $2 < q \leq 4$ Potts models were made and some of them substantiated by numerical simulations [11]. Analytical calculations concerning clusters occurring in systems of statistical physics and their boundaries, traditionally based on methods of conformal field theory and the Coulomb gas [12], have recently seen major advances from the insight that fractal random curves can be described in a framework dubbed *stochastic Loewner evolution* (SLE) [13, 14]. Collecting these observations, a more systematic analysis of the relation between the critical and tricritical branches of

the Potts model and their connection to the FK and geometric clusters has been performed, resulting in the identification of exact values for the different cluster fractal dimensions and their numerical verification [15, 16].

Coupling spin models to the planar random lattices of the dynamical triangulations model of two-dimensional Euclidean quantum gravity [17] corresponds to the introduction of a particular type of annealed connectivity disorder. The resulting randomness is strong, leading to a change in critical behaviour of virtually all types of coupled matter variables [18]. Geometrically, it is characterised by a large fractal dimension $d_h \approx 4$ of these lattices of topological dimension two. It is interesting to see how the relation between geometric and FK clusters of the Potts model works out far away from the regularity of a Bravais lattice. The mentioned relations between geometric and FK clusters have not yet been rigorously established, such that evidence from further models is highly welcome support. Besides, the behaviour of these fractal cluster objects on lattices which are themselves highly fractal is of particular interest in itself.

2 Dynamical triangulations and the KPZ formula

The dynamical triangulations approach provides a constructive model for Euclidean quantum gravity in general dimensions [17]. It regularizes the path integral over fluctuating metrics naturally occurring in an attempt to quantize the gravitational interaction by a sum over combinatorial manifolds realised as simplicial complexes [19]. In two dimensions, the resulting canonical ensemble of discrete surfaces can be defined as that of all possible gluings of a given number N_2 of equilateral triangles to a closed surface of fixed (usually spherical) topology, where all resulting triangulations are counted with the same weight in the partition sum. This is a purely combinatorial definition, and due to the equilaterality of the triangles the resulting triangulations cannot in general be embedded in the Euclidean \mathbb{R}^3 and, in fact, one is only interested in their *intrinsic* geometry. Many counting problems related to these graphs can be solved exactly by using matrix integrals or general combinatorial techniques [20]. For our purposes it is sufficient to note that the resulting random graphs are highly fractal with blobs (“baby universes”) of arbitrary size being connected to the main graph body with a minimal number of links. This structure entails an internal Hausdorff dimension $d_h = 4$ [17]. Decorating the vertices (or edges, faces) of the graphs with matter variables, corresponding to an annealed geometrical average, leads to a change of universality class at criticality, expressed through a dressing of the conformal weights Δ of the

matter fields given by the KPZ formula [18],

$$\tilde{\Delta} = \frac{\sqrt{1-c+24\Delta} - \sqrt{1-c}}{\sqrt{25-c} - \sqrt{1-c}}, \quad (1)$$

where c denotes the central charge of the coupled matter model. Analogously, the change of the string-susceptibility exponent γ_s can be expressed in terms of c [18].

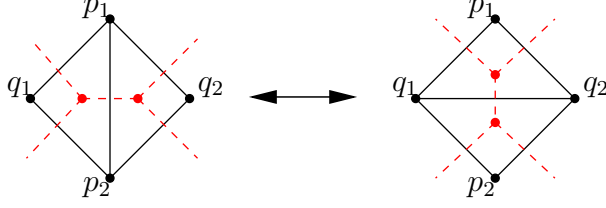
As mentioned above, the fractal properties of FK and geometric clusters of the square-lattice Ising model have been studied rather extensively [9, 10, 16]. Here, we are interested in the (normalised) fractal dimensions d_C^{FK}/d resp. d_C^{G}/d of the incipient percolating Fortuin-Kasteleyn resp. geometric cluster at criticality (where $d = 2$ denotes the spatial dimension). For the FK clusters, $d_C^{\text{FK}}/d = 1 - \beta/d\nu = 15/16$ is an exact result [21]. From the identification of the geometric Ising clusters with the FK clusters of a $q = 1$ tricritical Potts model, one finds $d_C^{\text{G}}/d = 187/192$ [10]. These values are related to the conformal weights Δ above as $\Delta_C = 1 - d_C/d$ [22]. To describe the coupling of the two-dimensional Ising model to quantum gravity, the resulting weights $\Delta_C^{\text{FK}} = 1/16$ and $\Delta_C^{\text{G}} = 5/192$ have to be dressed according to Eq. (1). Since for both, the Ising and $q = 1$ tricritical Potts models, the central charge $c = 1/2$, from (1) we find $\tilde{\Delta}_C^{\text{FK}} = 1/6$ and $\tilde{\Delta}_C^{\text{G}} = 1/12$. Therefore, we expect the following cluster fractal dimensions for the Ising model coupled to dynamical triangulations,

$$\tilde{d}_C^{\text{FK}}/d_h = 1 - \tilde{\Delta}_C^{\text{FK}} = 5/6, \quad \tilde{d}_C^{\text{G}}/d_h = 1 - \tilde{\Delta}_C^{\text{G}} = 11/12, \quad (2)$$

which now have been written in units of the Hausdorff dimension d_h of the graphs coupled to the spin model, the value of which is not known exactly but numerically found to be consistent with $d_h = 4$ for all $0 \leq c \leq 1$ [23, 24].

3 Numerical simulation method and results for the Ising model

For a numerical simulation of the system, due to the annealed nature of the disorder both, the underlying dynamical triangulations as well as the coupled Ising spins, must be updated in parallel. An ergodic set of Monte Carlo updates for the dynamical triangulations is given by the so-called Pachner moves [25]. For the canonical ensemble of a fixed number N_2 of triangles in two dimensions, these reduce to the following flip between two adjacent triangles,



where the dashed lines indicate the effect on the dual ϕ^3 graph. To improve the decorrelation of adjacent configurations in the Monte-Carlo Markov chain, this local update is complemented by intermittent non-local rewirings of the blob structure of the graph known as “baby-universe surgery method” [26, 27]. We consider the case of non-degenerate triangulations, corresponding to dual one-point irreducible ϕ^3 Feynman diagrams and place the Ising spins on the faces of the triangulation or, equivalently, the vertices of the dual ϕ^3 graphs. The coupled Ising model is being updated according to the Swendsen-Wang cluster algorithm [6] in order to alleviate the critical slowing down of the spin variables expected since simulations are performed at the exactly known inverse critical temperature $\beta_c = \frac{1}{2} \ln \frac{108}{23} \approx 0.7733$ of the coupled system [28].

Although the fractal dimensions d_C^{FK} and d_C^{G} could be determined directly by means of an appropriate geometrical analysis of the clusters, it is convenient to exploit relations to more easily accessible quantities. The fractal dimension of Fortuin-Kasteleyn clusters, d_C^{FK} , is related to the magnetic susceptibility exponent $\gamma = \gamma^{\text{FK}}$ as $d_C/d_h = \frac{1}{2}(1 + \gamma/d_h\nu)$ [29], where ν denotes the critical exponent of the correlation length. The susceptibility $\chi = \chi^{\text{FK}}$ can be sampled by the following estimator in the cluster language [30],

$$\chi \equiv \frac{1}{N_2} \left\langle \left(\sum_i \sigma_i \right)^2 \right\rangle = \frac{1}{N_2} \left\langle \sum_i |C_i|^2 \right\rangle, \quad (3)$$

where $\sigma_i = \pm 1$ denotes the Ising spins. The sum in the second expression is over all FK clusters C_i (including the percolating cluster as well as isolated sites) of a given configuration and $|C_i|$ denotes the number of spins in cluster i . Close to the critical point and for a system of finite size N_2 , standard finite-size scaling (FSS) arguments suggest that χ scales as

$$\chi \sim N_2^{\gamma/d_h\nu} = N_2^{2d_C/d_h-1}. \quad (4)$$

Thus, Eqs. (3) and (4) allow for a FSS determination of the fractal dimension d_C^{FK}/d_h of the FK clusters from information about the cluster distribution naturally produced by the Swendsen-Wang cluster update of the spins. In complete analogy, a “geometric susceptibility” χ^{G} can be estimated with C_i now denoting the *geometric* clusters in relation (3), and its FSS is described by Eq. (4) with exponent d_C^{G}/d_h .

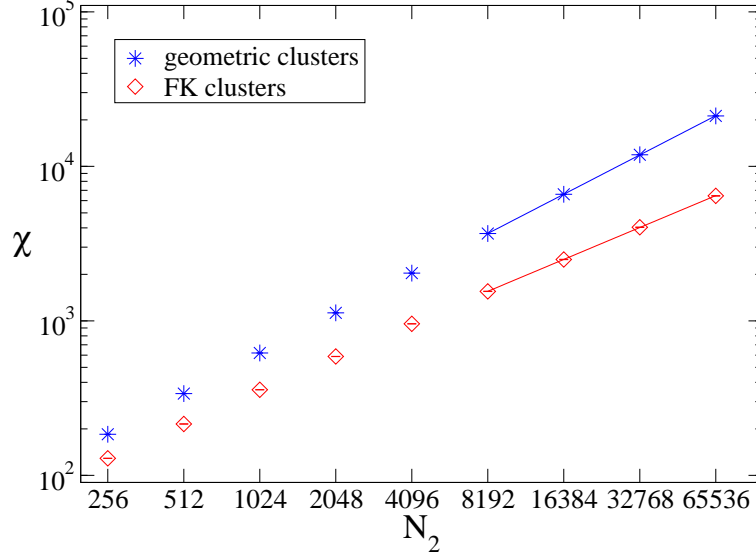


Fig. 1. Finite-size scaling of the magnetic (“FK clusters”) and geometric (“geometric clusters”) susceptibilities χ of the Ising model coupled to regular dynamical triangulations with N_2 triangles and at the critical inverse temperature $\beta_c = \frac{1}{2} \ln \frac{108}{23} \approx 0.7733$. The lines show fits of the functional form (4) to the data.

As a check and gauge for the method, we determined the cluster fractal dimensions for the Ising model on the square lattice, using $L \times L$ lattices with $L = 8, \dots, 512$. From fits of the form (4) to the (magnetic and geometric) susceptibility data, we find $d_C^{\text{FK}}/d = 0.93756(15)$ and $d_C^{\text{G}}/d = 0.97370(13)$ (including lattices of sizes $L = 16, \dots, 512$), in very good agreement with the exact results $d_C^{\text{FK}}/d = 15/16 \approx 0.93750$ and $d_C^{\text{G}}/d = 187/192 \approx 0.97396$. For the dynamical triangulations model, only graphs of somewhat smaller sizes can be equilibrated, and we consider volumes $N_2 = 256, 512, \dots, 65536$. Figure 1 shows the resulting susceptibility data in a doubly logarithmic plot together with fits of the form (4) to the data. Due to rather strong finite-size corrections, only the largest lattice sizes can be included in the uncorrected fit (4) with reasonable fit quality. For the range $N_2 = 8192, \dots, 65536$, we arrive at the estimates $\tilde{d}_C^{\text{FK}}/d_h = 0.84301(93)$ and $\tilde{d}_C^{\text{G}}/d_h = 0.92220(55)$, in marginal agreement with the conjectured values of Eq. (2), $\tilde{d}_C^{\text{FK}}/d_h \approx 0.83333$ and $\tilde{d}_C^{\text{G}}/d_h \approx 0.91667$. From previous experience with the dynamical triangulations model, one indeed expects very strong finite-size corrections to be present [24]. These result from the small effective linear extents of lattices of a given number of triangles reflected in the large fractal dimension $d_h \approx 4$. As dominant contribution, we expect analytic corrections to the effective linear extent,

$$L_{\text{eff}}(N_2) = L_0 N_2^{1/d_h} + L_1 + L_2 N_2^{-1/d_h} + \dots, \quad (5)$$

such that the susceptibility should scale as $\chi \sim [L_{\text{eff}}(N_2)]^{\gamma/\nu}$. Since the data are not precise enough to determine an additional exponent, we fix $d_h = 4$ in the expression for $L_{\text{eff}}(N_2)$. Already with one correction term only ($L_2 = 0$), the fit results,

$$\tilde{d}_C^{\text{FK}}/d_h = 0.8291(19), \quad \tilde{d}_C^{\text{G}}/d_h = 0.9132(12) \quad (6)$$

for the range $N_2 = 1024, \dots, 65536$, move considerably closer to the conjectured values. With both correction terms (i.e., variable L_2), we get full consistency with estimates $\tilde{d}_C^{\text{FK}}/d_h = 0.8349(118)$ and $\tilde{d}_C^{\text{G}}/d_h = 0.9164(68)$. Conversely, if we *fix* $\tilde{d}_C^{\text{FK}}/d_h$ resp. $\tilde{d}_C^{\text{G}}/d_h$ at the conjectured values, the fits with one correction term give reasonable fit qualities with $Q = 0.20$ resp. $Q = 0.03$, and very high-quality fits are reached when including both correction terms with $Q = 0.66$ resp. $Q = 0.42$.

4 General results for the Potts model

To understand the general relation between FK and geometric clusters and the critical and tricritical branches of the Potts model coupled to dynamical triangulations, we consider the following parametrisation of the critical q -state Potts model [31],

$$\sqrt{q} = -2 \cos(\pi/\kappa), \quad \kappa = \frac{1+m}{m}, \quad m = 1, 2, 3, \dots, \quad (7)$$

where $1 \leq \kappa \leq 2$ and the central charge $c = 1 - 6/m(m+1)$, such that $\kappa = 2, 3/2, 4/3, 6/5, 1$ corresponds to the $q = 0, 1, 2, 3, 4$ Potts models with $c = -2, 0, 1/2, 4/5, 1$, respectively. The conformal weights of the primary operators follow from the Kac table [22],

$$\Delta_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}, \quad r, s = 1, 2, \dots, m-1, \quad (8)$$

and they are identified with the physical operators of the Potts model as follows,

$$\begin{aligned}
\Delta_\epsilon &= \Delta_{2,1} = \frac{3\kappa}{4} - \frac{1}{2}, \\
\Delta_\sigma &= \Delta_{\frac{m+1}{2}-1, \frac{m+1}{2}} = \frac{1}{2} - \frac{3\kappa}{16} - \frac{1}{4\kappa}, \\
\Delta_C &= \Delta_{\frac{m+1}{2}, \frac{m+1}{2}} = \frac{1}{2} - \frac{3\kappa}{16} - \frac{1}{4\kappa}, \\
\Delta_H &= \Delta_{m,m} = \frac{1}{2} - \frac{\kappa}{4}, \\
\Delta_{EP} &= \Delta_{m+1, m+1} = \frac{1}{2} - \frac{1}{4\kappa}, \\
\Delta_{RB} &= \Delta_{m, m-1} = \frac{1}{2} + \frac{3}{4\kappa} - \frac{\kappa}{4},
\end{aligned} \tag{9}$$

where Δ_ϵ and $\Delta_\sigma = \Delta_C$ correspond to the leading energetic and magnetic operators, respectively, Δ_H denotes the weight corresponding to the cluster hull, Δ_{EP} the weight of the external perimeter and Δ_{RB} the weight of the “red bonds” of the cluster, cf. Ref. [16]. The various fractal dimensions of FK clusters are given by the corresponding renormalization-group eigenvalues, $d_\alpha = y_\alpha \equiv d_h(1 - \Delta_\alpha)$, where $\alpha = C, H, EP, RB$.

In Ref. [16] it was argued that in general the fractal dimensions of the geometric clusters of the critical Potts model are identical to the fractal dimensions of the *tricritical* Potts model of the same central charge, to be reached by performing the central-charge conserving “duality transformation” $\kappa \rightarrow 1/\kappa$ in Eq. (7). For the tricritical model, the leading conformal weights are identified as

$$\begin{aligned}
\Delta_\epsilon &= \Delta_{1,2} = \frac{3}{4\kappa} - \frac{1}{2}, \\
\Delta_\sigma &= \Delta_{\frac{m}{2}, \frac{m}{2}} = \frac{1}{2} - \frac{3}{16\kappa} - \frac{\kappa}{4}, \\
\Delta_C &= \Delta_{\frac{m}{2}, \frac{m}{2}} = \frac{1}{2} - \frac{3}{16\kappa} - \frac{\kappa}{4}, \\
\Delta_H &= \Delta_{m+1, m+1} = \frac{1}{2} - \frac{1}{4\kappa}, \\
\Delta_{EP} &= \Delta_{m, m} = \frac{1}{2} - \frac{\kappa}{4}, \\
\Delta_{RB} &= \Delta_{\frac{m+4}{2}, \frac{m+1}{2}} = \frac{1}{2} + \frac{3\kappa}{4} - \frac{1}{4\kappa},
\end{aligned} \tag{10}$$

which, as can be seen, directly follow from those of Eq. (9) by replacing $\kappa \rightarrow 1/\kappa$.

For the weights of the unitary minimal models of Eq. (8), the KPZ formula (1) can be written as

$$\tilde{\Delta}_{r,s} = \frac{1}{2}(1 - \kappa + |s - \kappa r|), \quad (11)$$

explicitly revealing that all weights of the minimal models dressed for the coupling to quantum gravity are rational numbers. Dressing the weights (9) of the *critical branch* yields

$$\begin{aligned} \tilde{\Delta}_\epsilon &= \frac{\kappa}{2}, & \tilde{\Delta}_\sigma &= \frac{1}{2} - \frac{\kappa}{4}, \\ \tilde{\Delta}_C &= \frac{1}{2} - \frac{\kappa}{4}, & \tilde{\Delta}_H &= 1 - \frac{\kappa}{2}, \\ \tilde{\Delta}_{EP} &= \frac{1}{2}, & \tilde{\Delta}_{RB} &= \frac{3}{2} - \frac{\kappa}{2}, \end{aligned} \quad (12)$$

whereas the weights (10) of the *tricritical branch* become

$$\begin{aligned} \tilde{\Delta}_\epsilon &= \frac{3}{2} - \kappa, & \tilde{\Delta}_\sigma &= \frac{3}{4} - \frac{\kappa}{2}, \\ \tilde{\Delta}_C &= \frac{3}{4} - \frac{\kappa}{2}, & \tilde{\Delta}_H &= \frac{1}{2}, \\ \tilde{\Delta}_{EP} &= 1 - \frac{\kappa}{2}, & \tilde{\Delta}_{RB} &= \frac{1}{2} + \frac{\kappa}{2}. \end{aligned} \quad (13)$$

Note that now, in contrast to the regular lattice case of Eqs. (9) and (10), all weights depend *linearly* on the parameter κ . The resulting FK and geometric cluster fractal dimensions for the Potts model coupled to dynamical triangulations are summarised in Table 1.

5 Conclusions

We have shown how the fractal dimensions associated with the geometric clusters of the Potts model coupled to the dynamical triangulations model of Euclidean quantum gravity in two dimensions can be inferred from a mapping to the tricritical branch of the Potts model. To allow for an application of the KPZ framework, the conformal weights associated to the fractal dimensions d_C of the cluster, d_H of the cluster hull, d_{EP} of the external perimeter and d_{RB} of the cluster red bonds have been identified from the Kac table. Lifting the corresponding weights of the critical and tricritical branches to the dynamical triangulations results in rational values for the fractal dimensions of FK and

Table 1

Cluster fractal dimensions of the q -state Potts model on dynamical triangulations. The values of $\tilde{d}_{\text{EP}}/d_h$ and $\tilde{d}_{\text{RB}}/d_h$ for the geometric clusters are only formal since, by definition, $\tilde{d}_{\text{EP}}/d_h \leq \tilde{d}_{\text{H}}/d_h$. Thus one has $\tilde{d}_{\text{EP}}^{\text{G}}/d_h = \tilde{d}_{\text{H}}^{\text{G}}/d_h$ and the geometric clusters are multiply connected.

q	Type	\tilde{d}_{C}/d_h	\tilde{d}_{H}/d_h	$\tilde{d}_{\text{EP}}/d_h$	$\tilde{d}_{\text{RB}}/d_h$
1	FK	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
	Geo	1	$\frac{1}{2}$	$(\frac{3}{4})$	$(-\frac{1}{4})$
2	FK	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
	Geo	$\frac{11}{12}$	$\frac{1}{2}$	$(\frac{2}{3})$	$(-\frac{1}{6})$
3	FK	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{1}{10}$
	Geo	$\frac{17}{20}$	$\frac{1}{2}$	$(\frac{3}{5})$	$(-\frac{1}{10})$
4	FK	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	0
	Geo	$\frac{3}{4}$	$\frac{1}{2}$	$(\frac{1}{2})$	0

geometric clusters for the $q = 0, 1, 2, 3$ and 4 state Potts models summarised in Table 1. The duality symmetry $\kappa \rightarrow 1/\kappa$ present between the critical and tricritical weights on regular lattices, Eqs. (9) and (10), is no longer present in the dressed weights of Eqs. (12) and (13). As a peculiarity, we note that the fractal dimension of the geometric cluster hulls is a constant $\tilde{d}_{\text{H}}^{\text{G}}/d_h = 1/2$ independent of q . A scaling or duality relation between the dimensions of the external perimeter and cluster hulls found valid for the regular case [32],

$$(d_{\text{EP}}/d - \frac{1}{2})(d_{\text{H}}/d - \frac{1}{2}) = \frac{1}{16}, \quad (14)$$

does not apply to the model coupled to quantum gravity, but instead one gets,

$$(\tilde{d}_{\text{EP}}/d_h - \frac{1}{2})(\tilde{d}_{\text{H}}/d_h - \frac{1}{2}) = 0. \quad (15)$$

Also, while the four fractal dimensions fulfil the identity $d_{\text{C}} - d_{\text{H}} = \frac{1}{4}(d_{\text{EP}} - d_{\text{RB}})$ for regular lattices [16], for the random-lattice case we instead find $\tilde{d}_{\text{C}} - \tilde{d}_{\text{H}} = \frac{1}{2}(\tilde{d}_{\text{EP}} - \tilde{d}_{\text{RB}})$. A couple of further similar scaling relations can be formulated.

Our simulation results for the case of the $q = 2$ Potts or Ising model coupled to dynamical triangulations show full consistency of the measured fractal dimensions of the Fortuin-Kasteleyn and geometric clusters with the predictions resulting from the KPZ mapping, confirming that the properties of the geometric Potts clusters on dynamical triangulations are described by the corresponding tricritical Potts model of the same central charge. Corrections to

scaling are found to be much stronger for the random-graph model than for the square-lattice simulations performed as a gauge. These corrections, known to result from the small effective linear extents of the graphs due to their large Hausdorff dimension $d_h \approx 4$ [24], have to be explicitly taken into account to find consistency with the scaling predictions and satisfactory quality of the fits.

It would be interesting to see whether, as expected, the fractal dimensions of the geometric clusters of the $q \neq 2$ Potts models coupled to dynamical triangulations follow the predictions summarised in Table 1, in particular for the case of $q = 4$, where the critical and tricritical branches coalesce and the KPZ mapping (1) becomes marginal due to central charge $c = 1$.

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